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On the inviscid instability of a circular jet with external flow

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The influence of an external flow velocity on the instability of a circular jet has been investigated by means of the inviscid linearized stability theory. The instability properties of spatially growing axisymmetric and first-order azimuthal disturbances show that the external flow inhibits the instability of the circular jet, but increases the unstable frequency range. Similarity considerations lead to the result that, in a first approximation, the disturbed flow field is independent of the external flow velocity, if the axial co-ordinate is contracted by a suitably chosen stretching factor and if the disturbance frequency is reduced by the same factor. It is concluded that the large-scale structure of jet turbulence is modified in the same manner by the external flow.

1. Introduction

For the prediction of jet noise from a jet engine in flight, it is necessary to know the influence of forward flight speed on the jet turbulence. In a co-ordinate system fixed to the moving engine, the jet velocity decays to a constant ambient velocity U_∞ which corresponds to the flight speed of the engine. The influence of this external flow on the jet turbulence is not yet fully understood. Since the vorticity contained in the initial shear layer of the jet is proportional to the velocity difference $\Delta U = U_j - U_\infty$, where U_j is the jet exit velocity, one would reasonably assume that the jet turbulence would scale with ΔU, but experimental results of Sarohia & Massier (1977) and of Tanna & Morris (1977) seem to indicate that this is not necessarily true.

There is, however, a wide agreement that the jet is stretched in the axial direction by the external flow. This phenomenon leads to a decreasing spreading angle of the jet mixing zone with increasing external flow velocity U_∞ as observed among others by Tanna & Morris (1977) and by Sarohia (1979). First theoretical results on this stretching effect for a circular jet have been obtained by Squire & Trouncer (1944). To take the jet stretching into account, it was therefore reasonable to assume that in a first approximation the flow field remains unaffected by the external flow velocity U_∞ in a co-ordinate system which is axially contracted by a constant stretching factor $\sigma(U_\infty, U_j)$. Ffowcs Williams (1963) used a stretching factor $\sigma = U_j/\Delta U$ in his calculation of the noise from moving jets. Unfortunately, his theoretical results did not agree with measured noise data. Therefore one cannot exclude as one possible reason his assumption on the stretching of the jet flow field. Recently, Michalke & Michel (1979) re-examined the problem. They also used a stretching model for the jet flow and turbulence by assuming a stretching factor $\sigma = 1+AU_\infty/\Delta U$ with a stretching parameter A in the order of 1–3. Surprisingly, good agreement was found between theoretical and experimental results for the radiated noise, when a stretching parameter $A = 2$ was used. If this agreement is not an accidental one, it should be expected that, in fact, the influence of the external flow velocity U_∞ on the jet turbulence should be almost eliminated, if the turbulence is considered in a co-ordinate system axially contracted by the stretching factor σ. This can surely be checked by experimental investigation.

On the other hand, the large-scale structure of jet turbulence, as first found by Mollo-Christensen (1967), Crow & Champagne (1971) and Fuchs (1972), seems to be well modelled by linear stability theory, as was shown by Michalke (1971), Chan (1974) and others. Hence it should also be possible to study the influence of an external flow on jet turbulence by means of linear stability theory. If an axially contracted co-ordinate system exists in which the jet turbulence is approximately independent of the external flow velocity U_∞, then the same should be true for the results of linear stability theory.

The aim of the present paper is therefore to investigate the instability of a circular jet with external flow by means of linearized stability theory and to look for any similarity condition which can, at least approximately, eliminate the U_∞-dependence of the results. For simplicity the flow is assumed to be isothermal with vanishing Mach number. Furthermore, viscous effects as discussed by Morris (1976) and effects due to the slowly divergent jet flow as investigated by Crighton & Gaster (1976) and Plaschko (1979) are neglected, too. It was found by Armstrong, Michalke & Fuchs (1977) and by Stromberg, McLaughlin & Troutt (1980) that the large-scale structure of turbulence in a circular jet is dominated by the axisymmetric and first-order azimuthal components of turbulence. Hence we restrict our stability calculation to these two components, too.

In §2 the disturbance equation and the basic jet velocity profile are given and the results of the static ($U_\infty = 0$) case are presented. In §3 the effect of the external flow velocity U_∞ on the instability properties of the jet is discussed. Finally, in §4 it is shown that, at least approximately, the results become independent of U_∞, if a stretching factor σ is introduced.

2. Solution of the instability problem

The basic jet flow field to be investigated is assumed as a locally parallel flow with $U(r)$ being the axial velocity component of the undisturbed jet flow. The Reynolds number of the flow is assumed to be large and the density to be constant. Then the instability properties can be derived by using the Euler equation for an incompressible fluid in a cylindrical co-ordinate system (x, r, ϕ), where the x-axis corresponds to the jet axis. A small pressure disturbance

$$p'(x,r,\phi,t) = \tilde{p}(r)\exp\left[i(\alpha x + m\phi - \beta t)\right], \tag{1}$$

and corresponding disturbances of the axial, radial and azimuthal velocity components are introduced in the linearized Euler and continuity equations. The resulting disturbance equation can be found, for instance, in the book of Betchov & Criminale (1967). For spatially growing disturbances the circular frequency β and the azimuthal

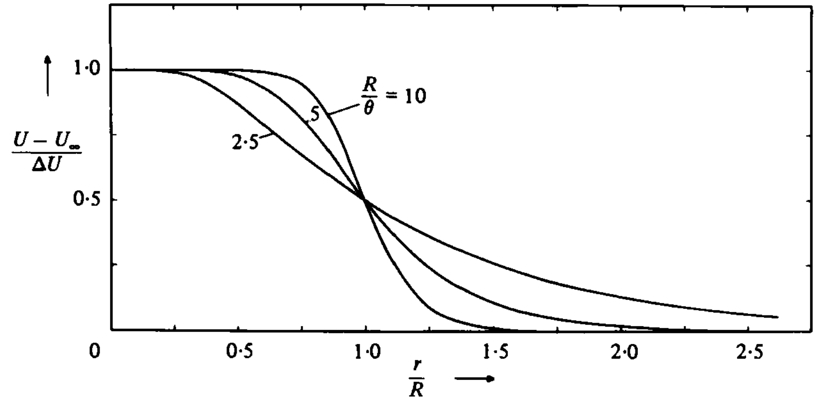

FIGURE 1. Normalized basic jet velocity profiles for various values of the jet parameter R/θ.

wavenumber m are real quantities, while $\alpha = \alpha_r + i\alpha_i$ is generally complex. α_r is the axial wavenumber and α_i the spatial growth rate. The disturbance is unstable if $-\alpha_i > 0$; $m = 0$ corresponds to an axisymmetric disturbance and $m = 1$ to the first azimuthal one.

If all other quantities are eliminated in favour of the pressure disturbance amplitude $\tilde{p}(r)$, then we obtain the disturbance equation:

$$\frac{d^2\tilde{p}}{dr^2}+\left[\frac{1}{r}-2\frac{dU}{dr}\Big/\left(U-\frac{\beta}{\alpha}\right)\right]\frac{d\tilde{p}}{dr}-\left[\alpha^2+\frac{m^2}{r^2}\right]\tilde{p} = 0. \tag{2}$$

The boundary conditions to be satisfied by the disturbance are:

$$\tilde{p}(0) = \text{finite}, \quad \tilde{p}(\infty) = 0. \tag{3}$$

Hence an eigenvalue problem is posed, namely that, for a given frequency β, the complex eigenvalue α has to be found which leads to an eigenfunction $\tilde{p}(r)$ of (2) satisfying the boundary conditions (3).

In the following we use the basic jet velocity profile

$$U(r) = \tfrac{1}{2}(U_j+U_\infty)-\tfrac{1}{2}(U_j-U_\infty)\tanh\left[\frac{1}{4}\frac{R}{\theta}\left(\frac{r}{R}-\frac{R}{r}\right)\right] = U_\infty+\Delta U\, U_0(r), \tag{4}$$

which, for the static case $U_\infty = 0$, has been investigated by Michalke (1971), Crighton & Gaster (1976), Morris (1976) and by Plaschko (1979) and which seems to model the circular jet flow in the potential-core region quite well, as was shown by Moore (1977). U_j is the jet core velocity, U_∞ the external flow velocity and $\Delta U = U_j - U_\infty$. θ is the momentum boundary layer thickness of the jet shear layer, defined by

$$\theta = \int_0^\infty \left[\frac{U-U_\infty}{U_j-U_\infty}\right]\left[1-\frac{U-U_\infty}{U_j-U_\infty}\right]dr. \tag{5}$$

The radius R denotes the middle of the jet shear layer, defined by $U(R) = \frac{1}{2}(U_j+U_\infty)$. The jet parameter R/θ characterizes jet velocity profiles at different axial positions. According to Crighton & Gaster (1976) we have, for $U_\infty = 0$,

$$R/\theta = 100/(3x/R+4). \tag{6}$$

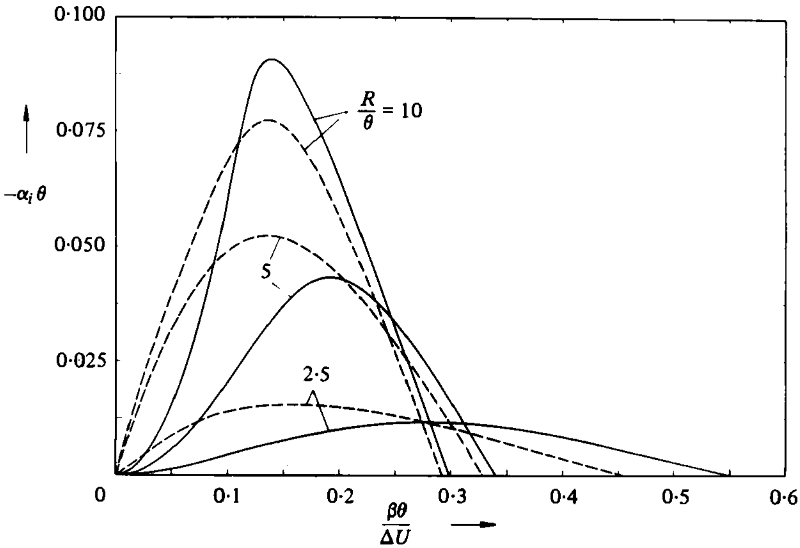

FIGURE 2. Spatial growth rate $-\alpha_i$ as function of the frequency β for the static case $U_\infty = 0$ and various values of the jet parameter R/θ. —, axisymmetric disturbance; ----, first azimuthal disturbance.

We restrict the present investigation to three values of the jet parameter, namely, $R/\theta = 10$, 5 and 2·5 which corresponds to the axial positions $x/(2R) = 1$, 2·67 and 6. In a normalized plot the velocity profiles are shown in figure 1.

The numerical procedure to solve the eigenvalue problem is as follows: the differential equation (2), written as a first-order system, has been integrated numerically by means of a Runge–Kutta procedure with automatic step-size choice and correction. The infinite integration region $0 < r < \infty$ is divided into two finite regions, an inner region $r_0 \leqslant r \leqslant r_m$ and an outer region $r_m \leqslant r \leqslant r_\infty$. Since dU/dr vanishes as $r \to 0$ and as $r \to \infty$, the asymptotic solutions of (2) satisfying the boundary condition (3) are

$$\tilde{p}^{(i)}_{(r)} \equiv C_1 f_i(r) \to C_1 I_m(\alpha r); \quad \tilde{p}^{(o)}_{(r)} \equiv C_2 f_o(r) \to C_2 K_m(\alpha r), \tag{7}$$

where I_m and K_m are the modified Bessel functions of order m. If r_0 and r_∞ (dependent on the jet parameter R/θ) are chosen such that dU/dr is sufficiently small, then we can start the integration with (7) at these points for a suitably chosen value α and integrate (2) up to $r = r_m$, which conveniently is chosen as $r_m = R$. The required matching conditions at $r = r_m$ are that $\tilde{p}$ and $d\tilde{p}/dr$ are to be continuous. Hence the eigenvalue equation for α is

$$F(\alpha) \equiv f_o(r_m)\, df_i(r_m)/dr - f_i(r_m)\, df_o(r_m)/dr = 0. \tag{8}$$

By means of a parabolic complex zero-search procedure, the eigenvalue α has been varied until $|F|$ was sufficiently small.

For the static case $U_\infty = 0$, the instability properties of the jet profile (4) have been already presented by Michalke (1971). It is convenient to normalize the spatial growth rate $-\alpha_i$ and the frequency β by means of the shear-layer thickness θ and the velocity difference ΔU, respectively, which is here identical with the jet velocity U_j. Figure 2 shows the dimensionless spatial growth rate $-\alpha_i\theta$ *vs.* the dimensionless frequency

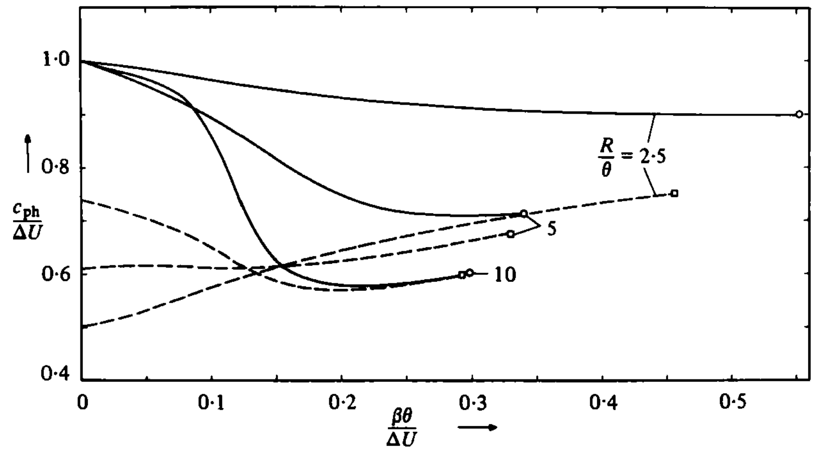

FIGURE 3. Phase velocity c_{ph} as function of the frequency β for the static case $U_\infty = 0$ and various values of the jet parameter R/θ. —, axisymmetric disturbance; ----, first azimuthal disturbance.

$\beta\theta/\Delta U$ for the jet parameter values $R/\theta = 10$, 5 and 2·5 and for the axisymmetric ($m = 0$, full line) and the first azimuthal ($m = 1$, broken line) disturbances. It is obvious that with decreasing R/θ, i.e. with increasing downstream distance x, the jet velocity profiles become less unstable. For low frequencies the first azimuthal disturbance is always more unstable than the axisymmetric one. We note, however, that for $R/\theta \geqslant 6·25$ approximately, i.e. $x/(2R) \leqslant 2$, maximum amplification is found for $m = 0$, cf. Michalke (1971), while further downstream the $m = 1$ disturbance is the most unstable one. This fact is in agreement with experimental results of Armstrong *et al.* (1977) who found strong first azimuthal components in jet turbulence for $x/D > 2$ and relevant Strouhal numbers $St = fD/\Delta U \approx 0·1$ to 0·6. Here D is the jet exit diameter and f the frequency. In fact, the peaks of the spatial growth rate belong to this Strouhal number range. If we put $D \simeq 2R$, we have peak Strouhal numbers for $m = 0$ of about 0·45, 0·30 and 0·22 for $R/\theta = 10$, 5 and 2·5, respectively. This corresponds to the fact that the peak of the turbulence spectrum is shifted to lower frequencies with increasing downstream distance.

In figure 3 the axial phase velocity $c_{ph} = \beta/\alpha_r$ is plotted as function of the frequency β for $m = 0$ (full lines) and $m = 1$ (broken lines) and for the same values of the jet parameter R/θ. The phase velocity of the axisymmetric disturbance ($m = 0$) always decreases from the jet velocity U_j at frequencies $\beta \to 0$ to lower values at higher frequencies. The lower the jet parameter R/θ, the higher is the phase velocity. For a fixed value of the jet parameter R/θ the phase velocity of the first azimuthal disturbance ($m = 1$) is always smaller than the axisymmetric one and increases for $R/\theta \leqslant 5$ at higher frequencies. The phase velocity measured in jet turbulence, excited artificially by a loudspeaker, by Crow & Champagne (1971), Chan (1974) and Bechert & Pfizenmaier (1975) showed the tendency of the axisymmetric disturbance, cf. Michalke (1971). For an unexcited jet, however, the measured phase velocity follows more closely the tendency of the first azimuthal disturbance at low frequencies, as can be seen from Ko & Davies (1975) and Armstrong (1977).

The difference in the frequency dependence of the phase velocity between that found in natural and excited jet turbulence at low frequencies can be explained by

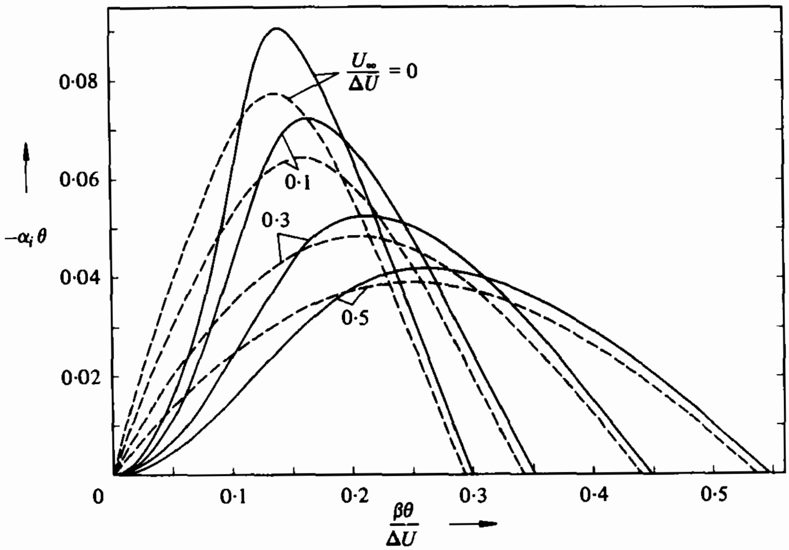

FIGURE 4. Spatial growth rate $-\alpha_i$ as function of the frequency β for the jet parameter $R/\theta = 10$ and various values of the velocity ratio $U_\infty/\Delta U$. —, axisymmetric disturbance; ----, first azimuthal disturbance.

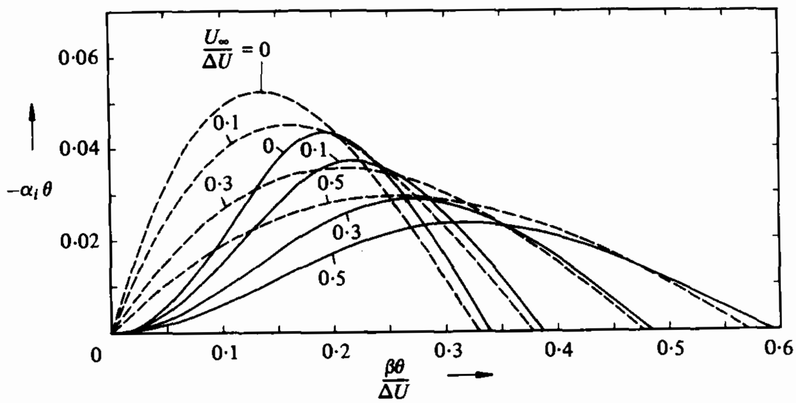

FIGURE 5. Spatial growth rate $-\alpha_i$ as function of the frequency β for the jet parameter $R/\theta = 5$ and various values of the velocity ratio $U_\infty/\Delta U$. —, axisymmetric disturbance; ----, first azimuthal disturbance.

means of the present results as follows. For low frequencies the natural jet turbulence has strong first azimuthal components sufficiently far downstream of the jet exit, as was found by Armstrong *et al.* (1977). From the view of stability theory this is quite reasonable, since at low frequencies the jet flow is more unstable to first azimuthal disturbances than to axisymmetric ones. Hence the phase velocity of natural jet turbulence will also be dominated by these components. If the jet turbulence is, however, excited by a loudspeaker, it is the axisymmetric disturbance which is excited most strongly, because of the axisymmetric sound field and initial conditions at the jet exit. It is clear that under these circumstances the measured phase velocity of the excited jet turbulences will have the tendency of that of the axisymmetric disturbance. These considerations show that excited jet turbulence will generally not

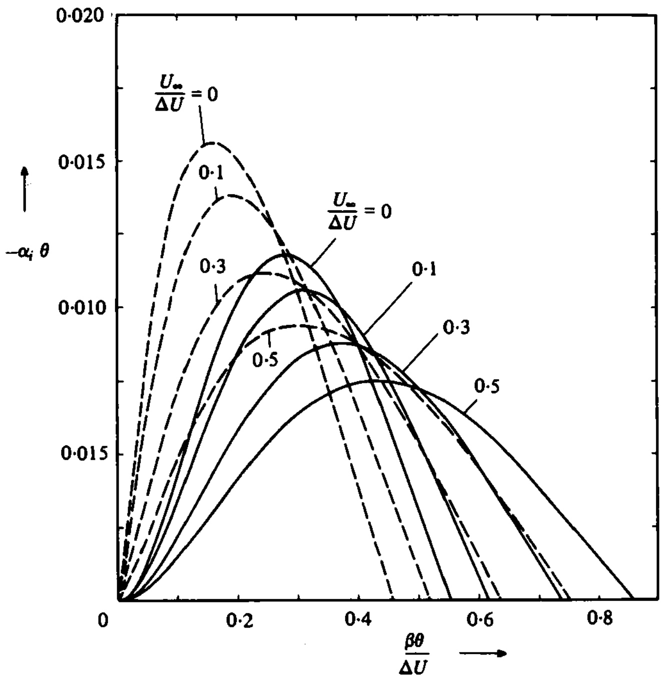

FIGURE 6. Spatial growth rate $-\alpha_i$ as function of the frequency β for the jet parameter $R/\theta = 2{\cdot}5$ and various values of the velocity ratio $U_\infty/\Delta U$. —, axisymmetric disturbance; ----, first azimuthal disturbance.

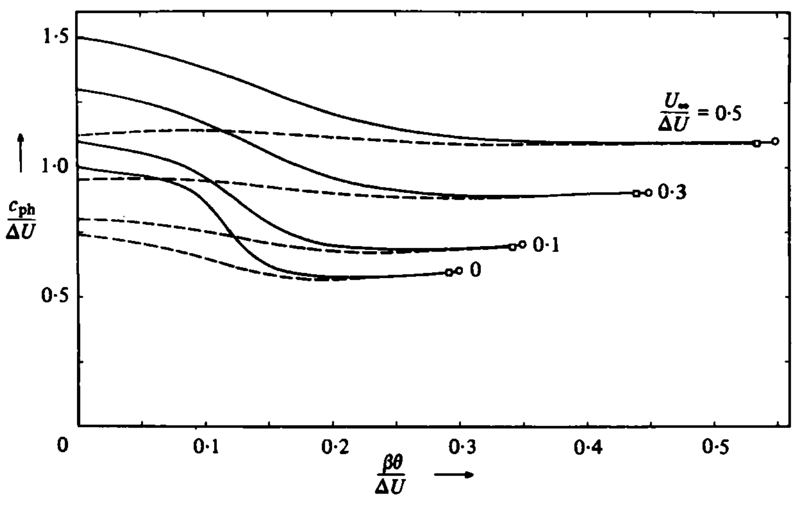

FIGURE 7. Phase velocity c_{ph} as function of the frequency β for the jet parameter $R/\theta = 10$ and various values of the velocity ratio $U_\infty/\Delta U$. —, axisymmetric disturbance; ----, first azimuthal disturbance.

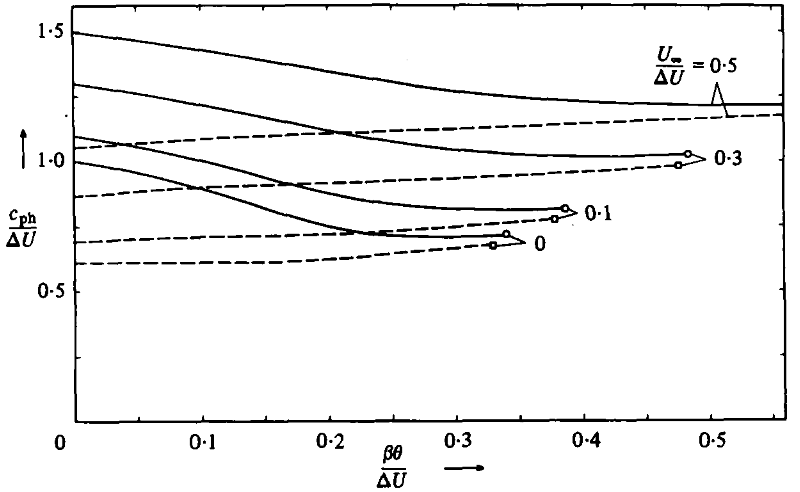

FIGURE 8. Phase velocity c_{ph} as function of the frequency β for the jet parameter $R/\theta = 5$ and various values of the velocity ratio $U_\infty/\Delta U$. —, axisymmetric disturbance; ----, first azimuthal disturbance.

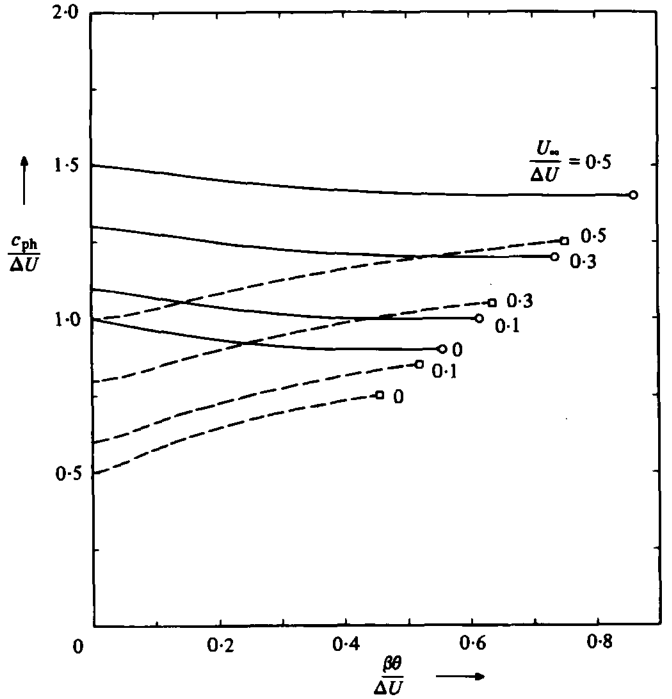

FIGURE 9. Phase velocity c_{ph} as function of the frequency β for the jet parameter $R/\theta = 2·5$ and various values of the velocity ratio $U_\infty/\Delta U$. —, axisymmetric disturbance; ----, first azimuthal disturbance.

have the properties of natural jet turbulence, unless all azimuthal components of natural jet turbulence are equally strongly amplified by the exciting sound field.

3. Influence of the external flow velocity on the jet instability

Further calculations have been made for the circular jet with the external flow velocity $U_\infty > 0$. The results are shown for the velocity ratios $U_\infty/\Delta U = 0$, 0·1, 0·3 and 0·5. In figures 4–6 the spatial growth rates *vs.* frequency of the axisymmetric (full lines) and first azimuthal (broken lines) disturbances are shown for the jet parameter values $R/\theta = 10$, 5 and 2·5, respectively. In figures 7–9 the corresponding phase velocities are shown.

The common tendency is that with increasing external flow velocity U_∞ the jet flow becomes less unstable, since the spatial growth rate $-\alpha_i$ decreases. However, the region of unstable frequencies is increased and the peak of the spatial growth rate is shifted to higher frequencies. With respect to jet turbulence one would therefore expect that the downstream development of turbulence is retarded by the external flow, but that the spectrum becomes broader and its peak is shifted to higher frequencies, provided the natural random excitation mechanism is unaffected by U_∞.

The phase velocity of the disturbances is always increased by the external flow velocity U_∞ and can well exceed the value of the velocity difference $\Delta U = U_j - U_\infty$.

4. Similarity considerations

From the disturbance differential equation (2) it follows that the basic jet velocity profile $U(r)$, which is here of the form given by (4), enters the problem only in the ratio

$$\frac{dU}{dr}\bigg/\left(U-\frac{\beta}{\alpha}\right) = \frac{dU_0}{dr}\bigg/[U_0+U_\infty/\Delta U-\beta/(\alpha\Delta U)]. \tag{9}$$

For the static case $U_\infty = 0$ and spatially growing disturbances the eigenvalues α are a complex function g of the real frequency β, i.e.

$$\alpha \equiv \alpha_r + i\alpha_i = g(\beta). \tag{10}$$

In order to eliminate the influence of U_∞, we can introduce a modified complex eigenvalue α_0 by putting

$$\beta/\alpha = \beta/\alpha_0 + U_\infty = (\beta/\alpha_0)\,\sigma_0, \tag{11}$$

where

$$\sigma_0 \equiv \sigma_{0r} + i\sigma_{0i} = 1+\frac{\alpha_0}{\beta}U_\infty = 1+\frac{U_\infty}{c_0}+i\frac{\alpha_{0i}}{\alpha_{0r}}\frac{U_\infty}{c_0}. \tag{12}$$

Here $c_0 = \beta/\alpha_{0r}$ is a phase velocity. Equation (11) then yields

$$\alpha = \alpha_0/\sigma_0. \tag{13}$$

Furthermore, equation (9) becomes

$$\frac{dU}{dr}\bigg/\left(U-\frac{\beta}{\alpha}\right) = \frac{dU_0}{dr}\bigg/[U_0-(\beta/\sigma_0)/((\alpha_0/\sigma_0)\,\Delta U)]. \tag{14}$$

Purely formally, the parameter $U_\infty/\Delta U$ is now eliminated and with (13) and (14) the differential equation (2) has the equivalent eigenvalue relation as (10) in the static case, namely

$$\alpha_0/\sigma_0 = g(\beta/\sigma_0). \tag{15}$$

The only difference from the static case ($U_\infty = 0$) is that here the argument β/σ_0 of

$m \backslash R/\theta$	10	5	2·5
0	1·66	1·40	1·11
1	1·67	1·48	1·33

TABLE 1. Stretching parameter A_n calculated from the neutral phase velocities.

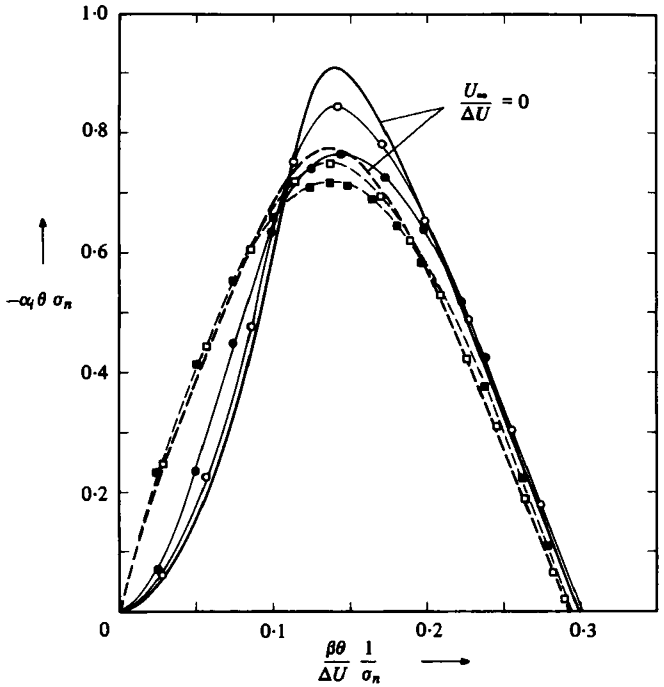

FIGURE 10. Reduced spatial growth rate $-\alpha_i \sigma_n$ *vs.* reduced frequency β/σ_n for the jet parameter $R/\theta = 10$ and the velocity ratios $U_\infty/\Delta U = 0{\cdot}1$ (○, □) and 0·5 (●, ■) compared with the static case $U_\infty = 0$. ○, ●, —, axisymmetric disturbance; □, ■, ----, first azimuthal disturbance.

the function g is now generally complex. An exception is the neutral case $\alpha_{0i} = 0$ where

$$\sigma_{0\text{neutral}} \equiv \sigma_n = 1 + U_\infty/c_n \tag{16}$$

is real with the neutral phase velocity c_n.

Let us now put with (12)

$$\Delta = \sigma_{0i}/\sigma_{0r} = \frac{\alpha_{0i}}{\alpha_{0r}} \frac{U_\infty}{c_0} \bigg/ (1 + U_\infty/c_0), \tag{17}$$

and assume $|\Delta| \ll 1$. Then we find from (12) that

$$\frac{1}{\sigma_0} \approx \frac{1}{\sigma_{0r}} [1 - i\Delta], \tag{18}$$

approximately, and can use the expansion

$$g(\beta/\sigma_0) \simeq g(\beta/\sigma_{0r}) - i(\beta/\sigma_{0r})\,\Delta g'(\beta/\sigma_{0r}) \simeq g(\beta/\sigma_{0r}) - i\Delta \frac{\alpha_{0r}}{\sigma_{0r}}, \tag{19}$$

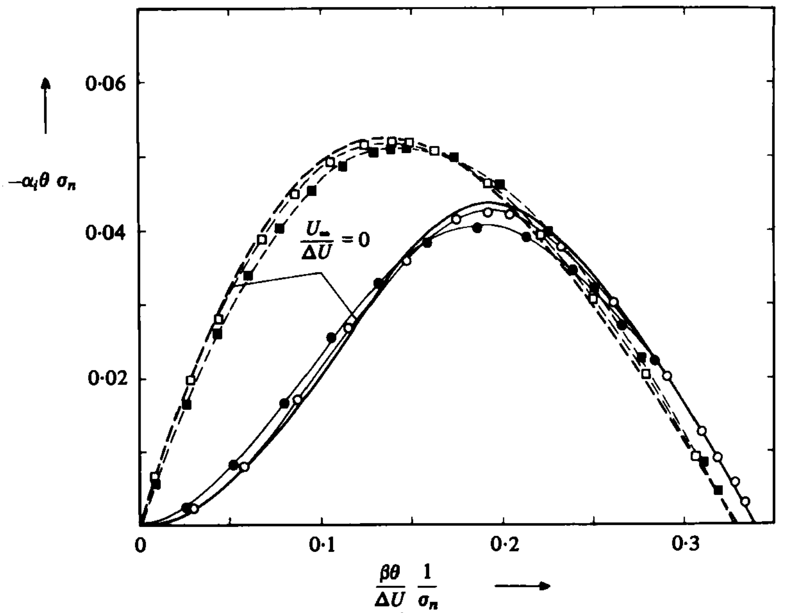

FIGURE 11. Reduced spatial growth rate $-\alpha_i\sigma_n$ *vs.* reduced frequency β/σ_n for the jet parameter $R/\theta = 5$ and the velocity ratios $U_\infty/\Delta U = 0{\cdot}1$ (○, □) and 0·5 (●, ■) compared with the static case $U_\infty = 0$. ○, ●, —, axisymmetric disturbance; □, ■, ----, first azimuthal disturbance.

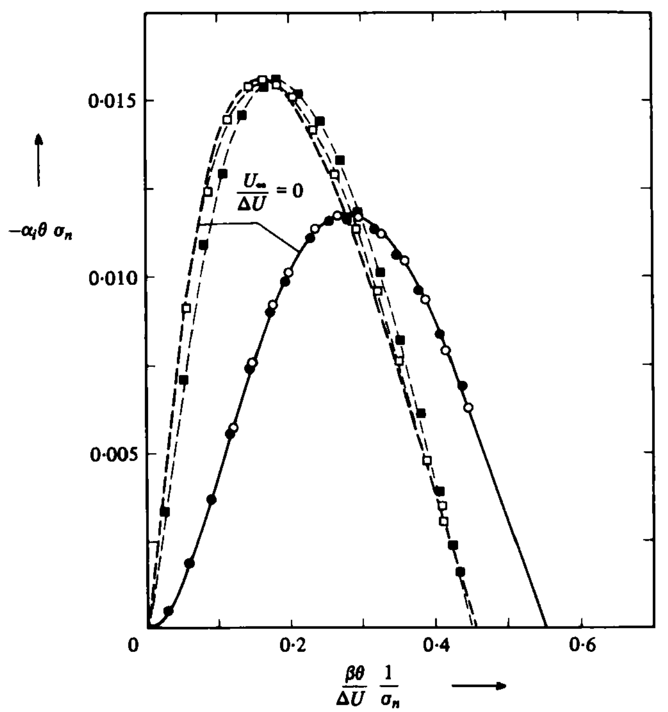

FIGURE 12. Reduced spatial growth rate $-\alpha_i\sigma_n$ *vs.* reduced frequency β/σ_n for the jet parameter $R/\theta = 2{\cdot}5$ and the velocity ratios $U_\infty/\Delta U = 0{\cdot}1$ (○, □) and 0·5 (●, ■) compared with the static case $U_\infty = 0$. ○, ●, —, axisymmetric disturbance; □, ■, ----, first azimuthal disturbance.

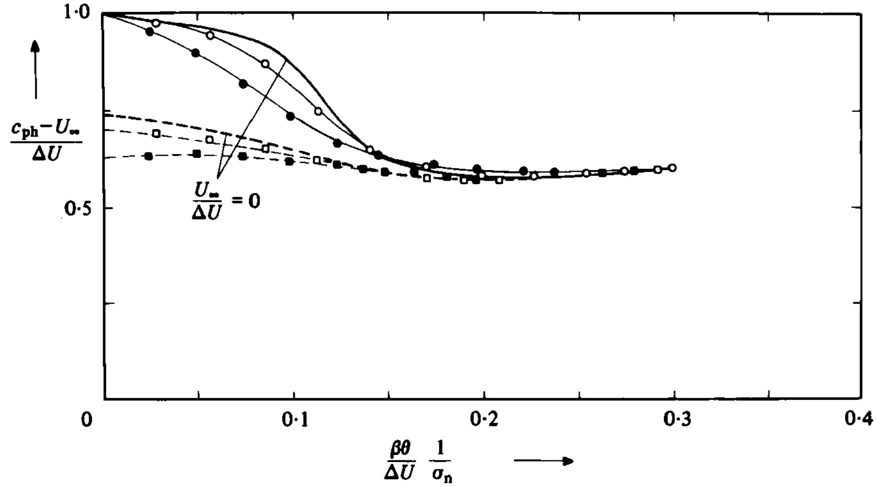

FIGURE 13. Relative phase velocity $c_{ph}-U_\infty$ *vs.* reduced frequency β/σ_n for the jet parameter $R/\theta = 10$ and the velocity ratios $U_\infty/\Delta U = 0{\cdot}1$ (○, □) and 0·5 (●, ■) compared with the static case $U_\infty = 0$. ○, ●, —, axisymmetric disturbance; □, ■, ----, first azimuthal disturbance.

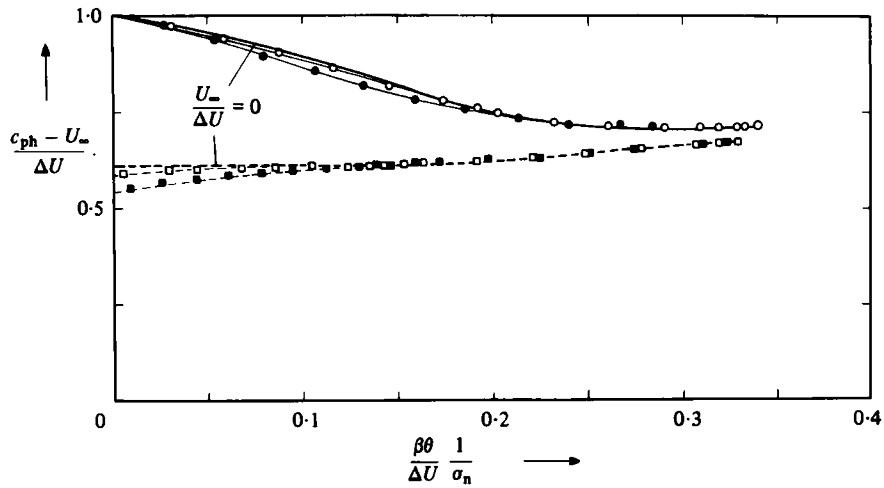

FIGURE 14. Relative phase velocity $c_{ph}-U_\infty$ *vs.* reduced frequency β/σ_n for the jet parameter $R/\theta = 5$ and the velocity ratios $U_\infty/\Delta U = 0{\cdot}1$ (○, □) and 0·5 (●, ■) compared with the static case $U_\infty = 0$. ○, ●, —, axisymmetric disturbance; □, ■, ----, first azimuthal disturbance.

where the approximation $g' = d(\alpha_0/\sigma_0)/d(\beta/\sigma_0) \simeq \alpha_{0r}/\beta$ has been used. Furthermore, from (13) there follows

$$\alpha = \alpha_0/\sigma_0 \simeq \alpha_0/\sigma_{0r}(1-i\Delta) \simeq \frac{\alpha_{0r}}{\sigma_{0r}} + i\left(\frac{\alpha_{0i}}{\sigma_{0r}} - \Delta\frac{\alpha_{0r}}{\sigma_{0r}}\right) = \frac{\alpha_{0r}}{\sigma_{0r}} + i\frac{\alpha_{0i}}{\sigma_{0r}^2}. \tag{20}$$

Here the definitions of σ_{0r}, σ_{0i} and Δ due to (12) and (17) have been used. Hence, with (19) and (20), it follows from (15) that, approximately,

$$(\alpha_{0r} + i\alpha_{0i})/\sigma_{0r} = g(\beta/\sigma_{0r}). \tag{21}$$

Since the argument β/σ_{0r} of the function g is now real, the complex eigenvalues α_0/σ_{0r} have the same dependence on β/σ_{0r} as α in the static case on β. Hence it follows

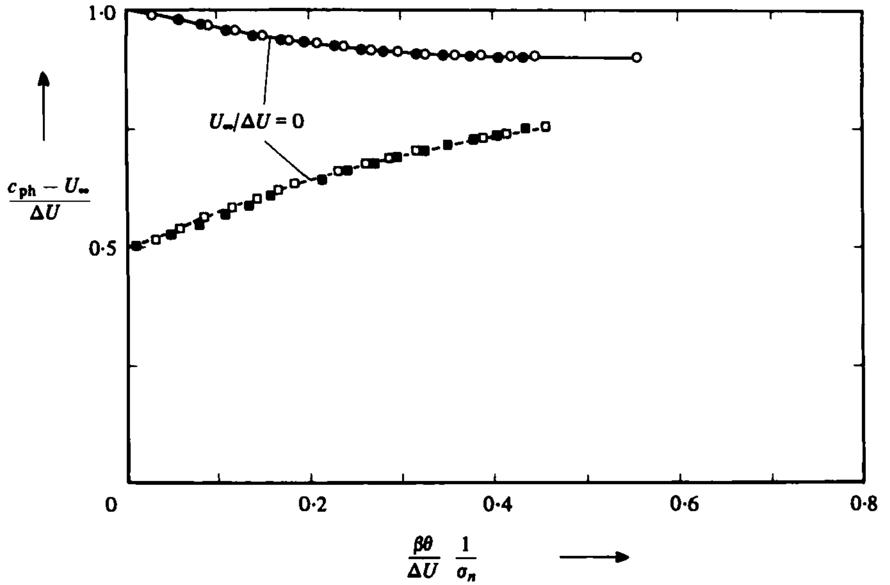

FIGURE 15. Relative phase velocity $c_{ph} - U_\infty$ *vs.* reduced frequency β/σ_n for the jet parameter $R/\theta = 2{\cdot}5$ and the velocity ratios $U_\infty/\Delta U = 0{\cdot}1$ (○, □) and 0·5 (●, ■) compared with the static case $U_\infty = 0$. ○, ●, —, axisymmetric disturbance; □, ■, ----, first azimuthal disturbance.

that the quantities α_{0r}/σ_{0r}, α_{0i}/σ_{0r} and $c_0 = \beta/\alpha_{0r}$ are functions of β/σ_{0r}, independent of U_∞ in this order of approximation. Furthermore, it follows from (20) and (21) that in this order of approximation the quantities

$$\alpha_r = \alpha_{0r}/\sigma_{0r}, \quad \sigma_{0r}\alpha_i = \alpha_{0i}/\sigma_{0r}, \quad c_{ph} - U_\infty \equiv \beta/\alpha_r - U_\infty = c_0, \tag{22}$$

would also depend only on β/σ_{0r}, regardless of the actual value of U_∞. Hence c_0 is the phase velocity of the static case ($U_\infty = 0$) which is a function of frequency, jet parameter R/θ and azimuthal order m.

In a next stage of approximation we may replace σ_{0r} by its neutral value σ_n due to (16), i.e. the phase velocity c_0 by the neutral one, c_n, which depends only on the jet parameter R/θ and on m. Then we can write σ_{0r} in the form of the stretching factor introduced by Michalke & Michel (1979):

$$\sigma_{0r} \approx \sigma_n = 1 + A_n U_\infty/\Delta U. \tag{23}$$

The values of $A_n = \Delta U/c_n$ evaluated from the neutral phase velocities of the static case are given in table 1.

In figures 10–12 the reduced spatial growth rates $(-\alpha_i\theta\sigma_n)$ *vs.* $\beta\theta/(\Delta U\sigma_n)$, for $m = 0$ and $m = 1$ and $U_\infty/\Delta U = 0{\cdot}1$ and 0·5, are compared with the static curves ($U_\infty = 0$) for the jet parameter values $R/\theta = 10$, 5 and 2·5, respectively. The calculated and reduced values for $U_\infty > 0$ are marked by circles ($m = 0$) and squares ($m = 1$), respectively. It is obvious that for $R/\theta = 10$ (figure 10) the correlation is not completely satisfactory, but that the deviation from the static curves becomes smaller for $R/\theta = 5$ (figure 11) and 2·5 (figure 12). This is reasonable, since for smaller values R/θ the ratio α_{0i}/α_{0r} contained in the quantity Δ due to (17) becomes smaller. In the neutral case the approximate relation (21) is exact. Hence for still smaller values of R/θ a still better agreement can be expected.

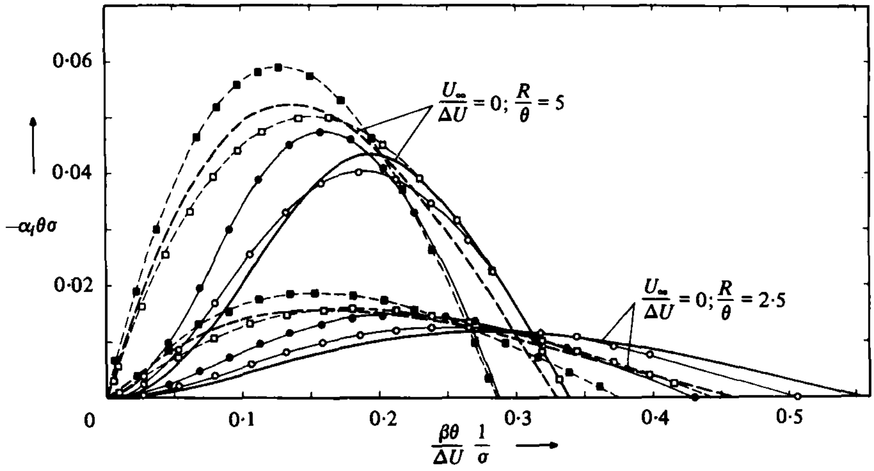

FIGURE 16. Spatial growth rate $-\alpha_i\sigma$ *vs.* frequency β/σ reduced by the universal stretching factor $\sigma = 1+AU_\infty/\Delta U$ for the jet parameter values $R/\theta = 5$ and 2·5 and the velocity ratio $U_\infty/\Delta U = 0{\cdot}5$ compared with the static case $U_\infty = 0$. ○, □, $A = 1{\cdot}4$; ●, ■, $A = 2$. ○, ●, —, axisymmetric disturbance; □, ■, ----, first azimuthal disturbance.

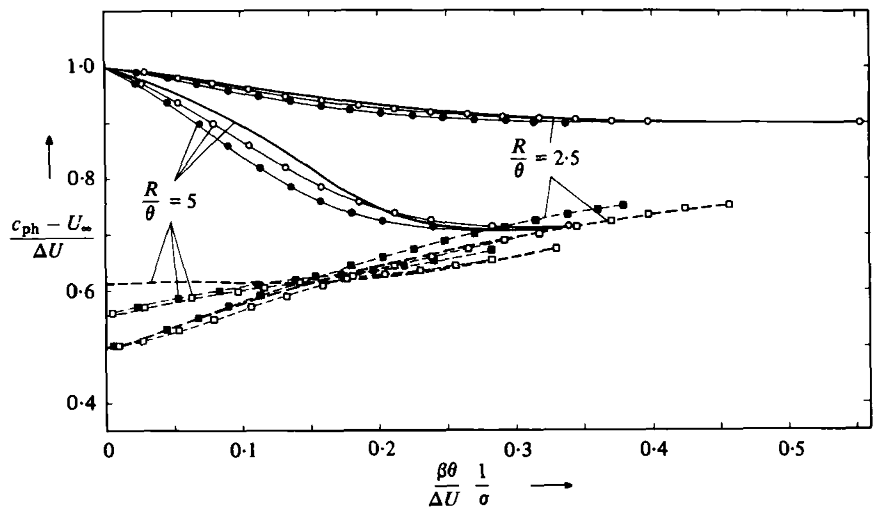

FIGURE 17. Relative phase velocity $c_{ph} - U_\infty$ *vs.* frequency β/σ reduced by the universal stretching factor $\sigma = 1+AU_\infty/\Delta U$ for the jet parameter values $R/\theta = 5$ and 2·5 and the velocity ratio $U_\infty/\Delta U = 0{\cdot}5$ compared with the static case $U_\infty = 0$. ○, □, $A = 1{\cdot}4$; ●, ■, $A = 2$. ○, ●, —, axisymmetric disturbance; □, ■, ----, first azimuthal disturbance.

The same tendency can be observed in figures 13–15, where the corresponding values of the reduced phase velocity $(c_{ph} - U_\infty)/\Delta U$ are plotted. For $R/\theta = 5$ and 2·5 the agreement is here more complete than with respect to the growth rates.

These results indicate that, at least approximately, similarity conditions exist. However, the stretching factor σ_n defined by (23) still depends on the azimuthal order m and on the downstream distance x via the jet parameter R/θ. In the paper of Michalke & Michel (1979) a universal stretching factor

$$\sigma = 1 + AU_\infty/\Delta U, \tag{24}$$

with a stretching parameter $A = 2$ has been found useful. From the present instability results one would prefer a value of about $A = 1{\cdot}4$ as being more reasonable, since it corresponds to $c_0/\Delta U = 0{\cdot}71$ which is a good mean value for the phase velocity. For $R/\theta = 5$ and $2{\cdot}5$, and for $U_\infty/\Delta U = 0{\cdot}5$, both values A have been checked. Figure 16 shows the results for the reduced growth rate and figure 17 those for the reduced phase velocity. Again, the calculated and reduced values for $U_\infty/\Delta U = 0{\cdot}5$ are marked by circles ($m = 0$) and squares ($m = 1$), respectively. It is obvious that the stretching parameter $A = 2$ leads to relatively unsatisfactory results, while $A = 1{\cdot}4$ gives acceptable agreement which should be still better for smaller values of the velocity ratio $U_\infty/\Delta U$. Therefore the stretching factor σ of (24) with $A = 1{\cdot}4$ seems to offer a suitable way to eliminate the influence of the external flow velocity U_∞ from the instability properties of the disturbed jet flow in a first approximation. Hence the same can be expected to be true for the large-scale structure of jet turbulence. If we now introduce a contracted axial co-ordinate $\xi = x/\sigma$, then the pressure disturbance according to (1) becomes

$$p'(\xi, r, \phi, t) = \tilde{p}(r)\, e^{-\alpha_i \sigma \xi} \exp\{i[m\phi + \beta(\xi/c_0 - t)]\}. \tag{25}$$

Since in the framework of this approximation the reduced spatial growth rate $-\alpha_i\sigma$ and the reduced phase velocity c_0 are independent of the external flow velocity U_∞ for constant $\beta/\sigma = \beta_0$, the pressure field remains unchanged in this contracted co-ordinate system (ξ, r, ϕ) provided the frequency β_0 in the static case is increased by a factor σ for $U_\infty > 0$. When, however, the instability properties of the jet can be described universally in this way, then one would expect that the turbulence generated by this instability would also be universal in a first approximation. If that were true we would find the turbulence spectra normalized by ΔU in the contracted co-ordinate system to be broader for $U_\infty > 0$ without any change in their magnitudes. The peak spectral value would be shifted to a frequency higher by the factor σ than that in the equivalent static case. There is some evidence from experimental results† that this hypothesis is not unrealistic. As a consequence it would, however, follow that the mean square values of turbulent quantities are increased by a factor σ in the presence of the external flow. This increase of turbulent energy at constant ΔU is due to the increased range of unstable frequencies, which leads to a broader spectrum. For the uncontracted, original co-ordinate system we should therefore expect that, for instance, the r.m.s. pressure $\tilde{p}(x, r, U_j, U_\infty)$ for the jet velocity U_j and external velocity U_∞ is related to the equivalent static r.m.s. pressure $\tilde{p}(x, r, U_e, 0)$ for an equivalent jet velocity U_e by

$$\frac{\tilde{p}^2(x, r, U_j, U_\infty)}{\tilde{p}^2(x, r, U_e, 0)} = \sigma\left(\frac{U_j - U_\infty}{U_e}\right)^4 \frac{F(x/\sigma, r)}{F(x, r)}, \tag{26}$$

where F is a universal function and σ is defined by (24) with $A \simeq 1{\cdot}4$. It is clear from (26) that the dependence of the mean square pressure ratio on the external flow velocity U_∞ is not altogether simple, if both measurements, for $U_\infty = 0$ and $U_\infty > 0$, are made at the same axial position x. For the plane mixing layer, where $A = 2$ is more appropriate, Yule (1972) found experimentally that the normalized turbulent intensity $q^2/\Delta U^2$ increases with increasing external flow velocity.

† Private communication from Dr U. Michel, Berlin.

5. Concluding remarks

The present investigation concerning the instability of a circular jet with external flow has shown that the spatial growth rate of axisymmetric and first azimuthal disturbances is reduced by the external flow, but the range of unstable frequencies and the phase velocity are increased. The influence of the external flow can be eliminated approximately, if the flow is considered in a co-ordinate system which is axially contracted by a stretching factor $\sigma = 1 + 1{\cdot}4U_\infty/\Delta U$ and if the frequency is reduced by the same factor σ. It is concluded that the same should be true for the jet turbulence as far as the large-scale structure is considered. Since the spectra should be broadened in presence of the external flow, the mean square values of turbulent quantities at fixed axial position x/σ should increase by the factor σ as compared to the case without external flow. The present results show that the assumptions concerning the influence of flight speed on jet turbulence made by Michalke & Michel (1979) for the prediction of jet noise in flight might require a correction. Their relation between the sound intensity $I_f(U_j, U_\infty)$ of the jet in flight and the sound intensity $I_0(U_e, 0)$ of the static jet with the effective static jet velocity $U_e = U_j - U_\infty$ is, for the noise radiation at an emission angle of 90° to the jet axis, given by

$$I_f(U_j, U_\infty) = \sigma^2 I_0(U_e, 0) \tag{27}$$

and was based on the assumption that in an axially contracted co-ordinate system the mean square value of the source quantity is independent of the flight speed U_∞. Good agreement with experimental results was found if the stretching parameter A of equation (24) was chosen as $A = 2$. In the light of the present results one should, however, expect a relation

$$I_f(U_j, U_\infty) = \sigma^3 I_0(U_e, 0) \tag{28}$$

with a stretching parameter $A \approx 1{\cdot}4$, instead of (27). For small values of $U_\infty/\Delta U$, however, the difference between (27) and (28) is very small, since we have in (27)

$$\sigma^2 = [1 + 2U_\infty/\Delta U]^2 \simeq 1 + 4U_\infty/\Delta U, \tag{29}$$

while (28) yields

$$\sigma^3 = [1 + 1{\cdot}4U_\infty/\Delta U]^3 \simeq 1 + 4{\cdot}2U_\infty/\Delta U. \tag{30}$$

This may explain why good agreement between experimental and theoretical results has been found by Michalke & Michel (1979), although the assumptions concerning the stretching mechanism of jet flow seem to be not quite correct.

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